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# A Hybrid Monte Carlo-Deterministic Nonlinear Second Moment Method for Gray Thermal Radiative Transfer with Implicit Emission-Absorption Physics and Explicit Material Properties

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## ABSTRACT

We propose a method for solving the frequency-integrated (gray) equations of thermal radiative transfer (TRT) with implicit emission-absorption physics and explicit material properties using Monte Carlo and deterministic techniques. Implicit emission-absorption physics is achieved by iterating to converge the nonlinear emission source term in the gray TRT equations. In the iteration, a Monte Carlo method solves a Boltzmann equation and a deterministic method solves a system of equations that do not depend on angle. We demonstrate our method on three problems: a method of manufactured solutions verification problem, an infinite medium noise study, and an inflow problem with a maximum-principle-preserving solution. The inflow problem demonstrates that the method can preserve a maximum principle even when the time step is large.

## KEYWORDS

Implicit emission-absorption, explicit material properties, Monte Carlo, TRT

## 1. Introduction

The thermal radiative transfer (TRT) equations are a system of nonlinear equations that can be difficult to solve numerically. Widely used solution strategies include deterministic methods that discretize the phase-space variables and Monte Carlo methods that sample them.

An important class of methods uses Fleck and Cummings' linearized TRT equations combined with Fleck and Cummings' Monte Carlo technique for solving the linearized equations ([Fleck and Cummings 1971](#)). The linearized equations are unconditionally stable, include semi-implicit emission-absorption physics, and recover the TRT equations as the size of the time step approaches zero. However, the linearized equations

admit solutions that violate a maximum principle when the time step is large. Violations can be eliminated with a sufficiently small time step, but numerical results indicate that the bound on the time step that guarantees that the solution satisfies the maximum principle is much smaller than the actual time step at which the violation occurs (Larsen and Mercier 1987).

Can Monte Carlo TRT methods preserve a maximum principle? The desire for Monte Carlo solutions that preserve a maximum principle motivates the search for more implicit methods. Brooks (1989) and N’Kaoua (1991) do not use Fleck and Cummings’ linearized equations and avoid Fleck and Cummings’ maximum principle violations by solving a nonlinear system of equations after tracking Monte Carlo particles with symbolic weights. Long et al. (2014) use Fleck and Cummings’ linearized equations, but they reduce the occurrence and severity of violations by iteratively updating the temperature used to compute the emission source in the linearized equations. Gentile and Yee (2016) do not use Fleck and Cummings’ linearized equations and avoid violations by iterating until the emission source in the TRT equations converges.

The method we present iterates to converge the emission source in the TRT equations, but with extra steps intended to improve iterative convergence, which Gentile and Yee found to be relatively slow in optically thick problems. Our approach resembles the method of Olivier et al. (2026), who iterate to converge the multigroup TRT emission source, but use a purely deterministic approach. The method we present is an extension of the linear transport method of Pozulp et al. (2026) to nonlinear transport. Pozulp et al. iterate to converge a linear scattering source, whereas we iterate to converge a nonlinear emission source.

Our method does not have fully implicit emission-absorption physics because we use constant material data within a time step: the temperature dependence of the heat capacity and absorption opacity is treated explicitly. However, by evaluating the temperature in the (frequency-integrated) Planck distribution using the end-of-time-step temperature, the emission-absorption physics provided by our gray TRT method is more implicit than the emission-absorption physics in methods which do not use the end-of-time-step temperature for this term.

The implicitness of our method provides opportunities to further increase the time-discretization accuracy. For example, our method could be extended to use 2nd-order time integration as in the deterministic method of Southworth et al. (2024), or extended to use implicit opacities as in the deterministic method of Olivier et al. (2025). Like Southworth et al. and Olivier et al., we couple a Boltzmann equation and a system of equations that do not depend on angle. The biggest difference between our work and theirs is that we use Monte Carlo and they do not. We intend our work to serve as a bridge by which Monte Carlo methods can attain some of the benefits of new developments in deterministic methods research.

The bridge that we construct is missing some important pieces. The frequency dependence of the TRT equations is neglected, and a severe noise issue prevents us from computing solutions to standard benchmarks, such as a gray Marshak wave or the Crooked Pipe problem of Graziani and LeBlanc (2000). The noise issue is left unresolved, though we make suggestions for how future work could fix the issue.

The Monte Carlo and deterministic components of the method are presented in the next section, after introducing the gray TRT equations and describing the multi-level iteration that we use to solve them. The formulation of the first hybrid nonlinear Second Moment method in Section 2 is a novel contribution. We use the term “hybrid” because our method employs Monte Carlo to solve the angle-dependent equation in the Second Moment method, and a deterministic method to solve the angle-independent

equations.

In [Section 3](#), we provide numerical results computed with our method. The three problems in [Section 3](#) are (i) a method of manufactured solutions verification problem, (ii) an infinite medium, and (iii) an inflow problem. The infinite medium highlights a noise issue in the method. The inflow problem demonstrates that the method can preserve a maximum principle even when the time step is large. In [Section 4](#), we conclude with a suggestion for how to fix the noise issue.

## 2. Numerical method

In this section, we write the equations that we wish to solve and then describe our novel procedure for computing the solution. The procedure involves an iteration in which both deterministic and Monte Carlo techniques are used. The Monte Carlo component of the method solves a Boltzmann equation and the deterministic component solves a system of equations that do not depend on angle. The iteration concludes when the nonlinear emission source term converges.

### 2.1. The gray thermal radiative transfer equations

The frequency-integrated (gray) thermal radiative transfer (TRT) equations are,

$$\frac{1}{c} \frac{\partial I}{\partial t} + \boldsymbol{\Omega} \cdot \nabla I + \sigma I = \frac{\sigma}{4\pi} B(T) + Q_r, \quad (1a)$$

$$C_v \frac{\partial T}{\partial t} = c\sigma E - \sigma B(T) + Q_m, \quad (1b)$$

where  $c$  is the speed of light,  $I = I(\mathbf{x}, \boldsymbol{\Omega}, t)$  is the radiation intensity,  $\mathbf{x} \in \mathbb{R}^{\dim}$  ( $\dim = 1, 2, 3$ ) is the spatial position coordinate,  $\boldsymbol{\Omega} \in \mathbb{S}^2$  is the unit direction vector,  $t$  is the time coordinate,  $\sigma$  is the absorption opacity,  $Q_r$  is an external radiation source,  $Q_m$  is an external material energy source,  $C_v$  is the heat capacity,  $T = T(\mathbf{x}, t)$  is the temperature,  $E$  is the radiation energy density,

$$E = \frac{1}{c} \int_{\mathbb{S}^2} I \, d\boldsymbol{\Omega},$$

and  $B(T)$  is a nonlinear function of the temperature,

$$B(T) = acT^4, \quad a = \frac{8\pi^5 k^4}{15h^3 c^3},$$

where  $h$  and  $k$  are Planck's constant and Boltzmann's constant, respectively. [Equations \(1a\)](#) and [\(1b\)](#) are subject to an inflow boundary condition which defines the radiation intensity on the domain boundary in the hemisphere of directions that point into the domain at a given location on the domain boundary surface,

$$I(\mathbf{x}, \boldsymbol{\Omega}, t) = I_{\text{inc}}(\mathbf{x}, \boldsymbol{\Omega}, t), \quad \mathbf{x} \in \partial\mathcal{D} \text{ and } \boldsymbol{\Omega} \cdot \mathbf{n} < 0, \quad (1c)$$

where  $I_{\text{inc}}$  is an arbitrary inflow function,  $\partial\mathcal{D}$  is the domain boundary, and  $\mathbf{n}$  is the outward-facing unit normal vector on the domain boundary surface. [Equations \(1a\)](#) and [\(1b\)](#) are also subject to initial conditions on  $I$  and  $T$ ,

$$I(\mathbf{x}, \boldsymbol{\Omega}, 0) = I_{\text{initial}}(\mathbf{x}, \boldsymbol{\Omega}), \quad (1d)$$

$$T(\mathbf{x}, 0) = T_{\text{initial}}(\mathbf{x}). \quad (1e)$$

## 2.2. The nonlinear Second Moment method

The Second Moment method of [Lewis and Miller \(1976\)](#) is an iterative method for linear transport that iterates to converge a linear scattering source. Following [Olivier et al. \(2026\)](#), we extend the Second Moment method to nonlinear transport by introducing a new iteration to converge the nonlinear emission source. Using the definition of the radiation energy density  $E$  from above and defining the radiation flux  $\mathbf{F}$ ,

$$E = \frac{1}{c} \int_{\mathbb{S}^2} I \, d\Omega, \quad (2)$$

$$\mathbf{F} = \int_{\mathbb{S}^2} \mathbf{\Omega} I \, d\Omega, \quad (3)$$

we can take the zeroth and first angular moments of [eq. \(1a\)](#) and then add and subtract the radiation energy density to obtain,

$$\frac{\partial E}{\partial t} + \nabla \cdot \mathbf{F} + c\sigma E = \sigma B(T) + Q_0, \quad \mathbf{x} \in \mathcal{D}, \quad (4a)$$

$$\frac{1}{c} \frac{\partial \mathbf{F}}{\partial t} + \frac{c}{3} \nabla E + \sigma \mathbf{F} = \mathbf{Q}_1 - \nabla \cdot \mathbf{R}, \quad \mathbf{x} \in \mathcal{D}, \quad (4b)$$

where  $\mathcal{D}$  is the spatial domain,  $Q_0$  and  $\mathbf{Q}_1$  are angular moments of  $Q_r$ ,

$$Q_0 = \int_{\mathbb{S}^2} Q_r \, d\Omega, \quad \mathbf{Q}_1 = \int_{\mathbb{S}^2} \mathbf{\Omega} Q_r \, d\Omega,$$

$\mathbf{R}$  is a functional of the intensity,

$$\mathbf{R} = \int_{\mathbb{S}^2} \left( \mathbf{\Omega} \otimes \mathbf{\Omega} - \frac{1}{3} \mathbb{I} \right) I \, d\Omega,$$

and  $\mathbb{I}$  is the identity tensor. Manipulating partial currents to derive a boundary condition for [eqs. \(4a\)](#) and [\(4b\)](#) results in,

$$\mathbf{F} \cdot \mathbf{n} = \frac{c}{2} E + 2F_{\text{in}} + \beta, \quad \mathbf{x} \in \partial\mathcal{D}, \quad (4c)$$

where  $F_{\text{in}}$  is a functional of the inflow data,

$$F_{\text{in}} = \int_{\mathbf{\Omega} \cdot \mathbf{n} < 0} \mathbf{\Omega} \cdot \mathbf{n} I_{\text{inc}} \, d\Omega,$$

and  $\beta$  is a functional of the intensity,

$$\beta = \int_{\mathbb{S}^2} \left( |\mathbf{\Omega} \cdot \mathbf{n}| - \frac{1}{2} \right) I \, d\Omega.$$

[Equations \(4a\)](#) and [\(4b\)](#) are also subject to initial conditions on  $E$  and  $\mathbf{F}$ ,

$$E(\mathbf{x}, 0) = E_{\text{initial}}(\mathbf{x}) = \frac{1}{c} \int_{\mathbb{S}^2} I_{\text{initial}} \, d\Omega, \quad (4d)$$

$$\mathbf{F}(\mathbf{x}, 0) = \mathbf{F}_{\text{initial}}(\mathbf{x}) = \int_{\mathbb{S}^2} \mathbf{\Omega} I_{\text{initial}} \, d\Omega. \quad (4e)$$

If  $\mathbf{R}$  and  $\beta$  are provided as data, then the system of [eqs. \(4a\)](#) to [\(4e\)](#) together with the material energy balance [eq. \(1b\)](#) is an equivalent reformulation of the gray TRT equations. That is, the  $E$  and  $\mathbf{F}$  that satisfy [eqs. \(4a\)](#) to [\(4e\)](#) and [eq. \(1b\)](#) are equal to the  $E$  and  $\mathbf{F}$  defined in [eqs. \(2\)](#) and [\(3\)](#), which are the zeroth and first angular moments of the radiation intensity, respectively. No approximations have been made.

The quantities  $\mathbf{R}$  and  $\beta$  are sometimes referred to as the transport correction tensor and boundary correction factor, respectively, because both are zero when the intensity is no more than linearly anisotropic. When  $\beta = 0$ , eq. (4c) simplifies to the Marshak boundary condition.

The equations of the gray TRT Second Moment method are,

$$\frac{1}{c} \frac{\partial I}{\partial t} + \mathbf{\Omega} \cdot \nabla I + \sigma I = \frac{\sigma}{4\pi} B(T) + Q_r, \quad (5a)$$

$$I(\mathbf{x}, \mathbf{\Omega}, t) = I_{\text{inc}}(\mathbf{x}, \mathbf{\Omega}, t), \quad \mathbf{x} \in \partial\mathcal{D} \text{ and } \mathbf{\Omega} \cdot \mathbf{n} < 0, \quad (5b)$$

$$\frac{\partial E}{\partial t} + \nabla \cdot \mathbf{F} + c\sigma E = \sigma B(T) + Q_0, \quad \mathbf{x} \in \mathcal{D}, \quad (5c)$$

$$\frac{1}{c} \frac{\partial \mathbf{F}}{\partial t} + \frac{c}{3} \nabla E + \sigma \mathbf{F} = \mathbf{Q}_1 - \nabla \cdot \mathbf{R}, \quad \mathbf{x} \in \mathcal{D}, \quad (5d)$$

$$\mathbf{F} \cdot \mathbf{n} = \frac{c}{2} E + 2F_{\text{in}} + \beta, \quad \mathbf{x} \in \partial\mathcal{D}, \quad (5e)$$

$$C_v \frac{\partial T}{\partial t} = c\sigma E - \sigma B(T) + Q_m, \quad \mathbf{x} \in \mathcal{D}, \quad (5f)$$

subject to the initial conditions in eqs. (1d), (1e), (4d) and (4e).

### 2.3. Time-discretized gray TRT Second Moment equations

Let  $\Delta t$  be the time step size and use the superscripts  $k$  and  $k+1$  to denote beginning-of-time-step and end-of-time-step quantities, respectively. The choice of whether to evaluate  $I, T, E$ , and  $\mathbf{F}$  at the beginning of the time step ( $I^k, T^k, E^k, \mathbf{F}^k$ ), the end of the time step ( $I^{k+1}, T^{k+1}, E^{k+1}, \mathbf{F}^{k+1}$ ), or somewhere in between determines the degree of implicitness of the method. By choosing end-of-time-step for the temperature in  $B(T)$  and beginning-of-time-step for the temperature in  $\sigma$  and  $C_v$ , we get semi-discrete equations with implicit emission-absorption physics and explicit material properties,

$$\frac{I^{k+1} - I^k}{c\Delta t} + \mathbf{\Omega} \cdot \nabla I^{k+1} + \sigma I^{k+1} = \frac{\sigma}{4\pi} B(T^{k+1}) + Q_r, \quad (6a)$$

$$I(\mathbf{x}, \mathbf{\Omega}, t) = I_{\text{inc}}(\mathbf{x}, \mathbf{\Omega}, t), \quad \mathbf{x} \in \partial\mathcal{D} \text{ and } \mathbf{\Omega} \cdot \mathbf{n} < 0, \quad (6b)$$

$$\frac{E^{k+1} - E^k}{\Delta t} + \nabla \cdot \mathbf{F}^{k+1} + c\sigma E^{k+1} = \sigma B(T^{k+1}) + Q_0^{k+1}, \quad \mathbf{x} \in \mathcal{D}, \quad (6c)$$

$$\frac{\mathbf{F}^{k+1} - \mathbf{F}^k}{c\Delta t} + \frac{c}{3} \nabla E^{k+1} + \sigma \mathbf{F}^{k+1} = \mathbf{Q}_1^{k+1} - \nabla \cdot \mathbf{R}^{k+1}, \quad \mathbf{x} \in \mathcal{D}, \quad (6d)$$

$$\mathbf{F}^{k+1} \cdot \mathbf{n} = \frac{c}{2} E^{k+1} + 2F_{\text{in}}^{k+1} + \beta^{k+1}, \quad \mathbf{x} \in \partial\mathcal{D}, \quad (6e)$$

$$\frac{C_v}{\Delta t} (T^{k+1} - T^k) = c\sigma E^{k+1} - \sigma B(T^{k+1}) + Q_m^{k+1}, \quad \mathbf{x} \in \mathcal{D}, \quad (6f)$$

where we omitted the  $k$  superscript on  $\sigma$  and  $C_v$  for brevity (the time-explicit material data  $\sigma^k = \sigma(T^k)$  and  $C_v^k = C_v(T^k)$  are written as  $\sigma$  and  $C_v$  in the equations above). Additionally,  $Q_r = Q_r(\mathbf{x}, \mathbf{\Omega}, t)$  and  $I_{\text{inc}}(\mathbf{x}, \mathbf{\Omega}, t)$  do not have superscripts because they are sampled continuously in time using Monte Carlo.

The emission-absorption terms are the terms in eqs. (6a), (6c) and (6f) that include  $B(T^{k+1})$ . Evaluating these terms at the end-of-time-step temperature  $T^{k+1}$  provides

implicit emission-absorption physics in problems with constant material data. However, heat capacity  $C_v$  varies with temperature, and absorption opacity  $\sigma$  is strongly temperature-dependent. Evaluating  $C_v$  and  $\sigma$  at the beginning-of-time-step temperature simplifies our semi-discrete equations at the cost of reduced implicitness, though implicit heat capacity and opacities are rarely used in TRT methods. It may be possible to use implicit opacities in our method by adopting a similar approach to [Olivier et al. \(2025\)](#), who demonstrated implicit opacities in a deterministic TRT method, but we have not investigated this.

Dropping the superscript on  $(\cdot)^{k+1}$  and replacing  $(\cdot)^k$  with  $(\cdot)^*$  allows us to rewrite the semi-discrete equations more compactly,

$$\boldsymbol{\Omega} \cdot \nabla I + \tilde{\sigma} I = \frac{\sigma}{4\pi} B(T) + Q_r + \frac{1}{c\Delta t} I^*, \quad (7a)$$

$$I(\mathbf{x}, \boldsymbol{\Omega}) = I_{\text{inc}}(\mathbf{x}, \boldsymbol{\Omega}), \quad \mathbf{x} \in \partial\mathcal{D} \text{ and } \boldsymbol{\Omega} \cdot \mathbf{n} < 0, \quad (7b)$$

$$\nabla \cdot \mathbf{F} + c\tilde{\sigma} E = \sigma B(T) + Q_0 + \frac{1}{\Delta t} E^*, \quad \mathbf{x} \in \mathcal{D}, \quad (7c)$$

$$\frac{c}{3} \nabla E + \tilde{\sigma} \mathbf{F} = \mathbf{Q}_1 - \nabla \cdot \mathbf{R} + \frac{1}{c\Delta t} \mathbf{F}^*, \quad \mathbf{x} \in \mathcal{D}, \quad (7d)$$

$$\mathbf{F} \cdot \mathbf{n} = \frac{c}{2} E + 2F_{\text{in}} + \beta, \quad \mathbf{x} \in \partial\mathcal{D}, \quad (7e)$$

$$\left( \frac{C_v}{\Delta t} + \sigma B(\cdot) \right) T = c\sigma E + Q_m + \frac{C_v}{\Delta t} T^*, \quad \mathbf{x} \in \mathcal{D}, \quad (7f)$$

where we moved the beginning-of-time-step sources to the right-hand side and defined  $\tilde{\sigma}$  to be,

$$\tilde{\sigma} = \sigma + \frac{1}{c\Delta t}.$$

For notational consistency with the other terms in [eqs. \(7a\) to \(7f\)](#), we write  $Q_r$  and  $I_{\text{inc}}$  without time dependence, even though we evaluate them continuously in time (using Monte Carlo sampling). [Figure 1](#) shows the gray TRT Second Moment outer iteration. In our hybrid Monte Carlo-deterministic approach, we use Monte Carlo to solve the left side of [Figure 1](#) and compute  $\mathbf{R}$  and  $\beta$  as Monte Carlo estimators. We then use a deterministic method to solve the angle-independent system of equations on the right side of [Figure 1](#) for  $T$ . We use  $T$  to compute the emission source, which we converge in an iteration.

## 2.4. Deterministic component of the hybrid method

We use a deterministic method to solve the nonlinear system of semi-discrete equations on the right side of [Figure 1](#). The semi-discrete equations depend only on space. We discretize these equations in space using finite elements.

### 2.4.1. Discrete system of angle-independent equations

The weak form of the material energy balance [eq. \(7f\)](#) is: find  $T \in Y_0$  such that,

$$\int u \left( \frac{C_v}{\Delta t} + \sigma B(\cdot) \right) T \, d\mathbf{x} = c \int u \sigma E \, d\mathbf{x} + \int u Q_m \, d\mathbf{x} + \int u \frac{C_v}{\Delta t} T^* \, d\mathbf{x}, \quad \forall u \in Y_0, \quad (8a)$$

and the weak form of eqs. (7c) and (7d) is: find  $(E, \mathbf{F}) \in Y_0 \times RT_0$  such that,

$$\int u \nabla \cdot \mathbf{F} \, d\mathbf{x} + c \int u \tilde{\sigma} E \, d\mathbf{x} = \int u \sigma B(T) \, d\mathbf{x} + \int u Q_0 \, d\mathbf{x} + \int \frac{1}{\Delta t} u E^* \, d\mathbf{x}, \quad \forall u \in Y_0, \quad (8b)$$

$$-\frac{c}{3} \int \nabla \cdot \mathbf{v} E \, d\mathbf{x} + \frac{2}{3} \int_{\Gamma_b} (\mathbf{v} \cdot \mathbf{n})(\mathbf{F} \cdot \mathbf{n}) \, ds + \int \tilde{\sigma} \mathbf{v} \cdot \mathbf{F} \, d\mathbf{x} = \mathcal{S}, \quad \forall \mathbf{v} \in RT_0, \quad (8c)$$

where the source term  $\mathcal{S}$  is,

$$\begin{aligned} \mathcal{S} = & \int \mathbf{v} \cdot \mathbf{Q}_1 \, d\mathbf{x} - \int_{\Gamma_b} \mathbf{v} \cdot \mathbf{Rn} \, ds + \frac{2}{3} \int_{\Gamma_b} (\mathbf{v} \cdot \mathbf{n})(2F_{\text{in}} + \beta) \, ds \\ & - \int_{\Gamma_0} \llbracket \mathbf{v} \rrbracket \cdot \llbracket \mathbf{Rn} \rrbracket \, ds + \int \nabla_h \mathbf{v} : \mathbf{R} \, d\mathbf{x} + \int \frac{1}{c\Delta t} \mathbf{v} \cdot \mathbf{F}^* \, d\mathbf{x}. \end{aligned} \quad (8d)$$

In eqs. (8a) to (8d),  $u$  and  $\mathbf{v}$  are test functions,  $E, \mathbf{F}$ , and  $T$  are trial functions,  $\Gamma_0$  is the set of unique faces on the interior of the mesh,  $\Gamma_b$  is the set of unique faces on the boundary of the mesh,  $Y_0$  is the degree-0 Discontinuous Galerkin finite element space,  $RT_0$  is the degree-0 Raviart-Thomas finite element space (Raviart and Thomas 1977, 1979). The “broken” gradient  $\nabla_h$ , the jump operator  $\llbracket \cdot \rrbracket$ , and the average operator  $\{\!\!\{ \cdot \}\!\!\}$  are defined as,

$$(\nabla_h u)|_K = \nabla(u|_K) \quad \forall K \in \mathcal{T}, \quad (9)$$

$$\llbracket u \rrbracket = \begin{cases} u_1 - u_2 & \mathcal{F} \in \Gamma_0 \\ u & \mathcal{F} \in \Gamma_b \end{cases}, \quad (10)$$

$$\{\!\!\{ u \}\!\!\} = \begin{cases} \frac{u_1 + u_2}{2} & \mathcal{F} \in \Gamma_0 \\ u & \mathcal{F} \in \Gamma_b \end{cases}, \quad (11)$$

where  $K$  is an element in the mesh,  $\mathcal{T}$  is the set of all mesh elements,  $\mathcal{F}$  is a mesh element face,  $u_1$  is the value of  $u$  on one side of the face, and  $u_2$  is the value of  $u$  on the other side.

#### 2.4.2. Newton-like solve of the angle-independent system

Equations (8a) to (8c) are a discrete system of nonlinear equations of the form,

$$\begin{bmatrix} \mathbf{M}_F & -\frac{c}{3} \mathbf{D}^T \\ \mathbf{D} & \mathbf{M}_E \\ & -\mathbf{M}_a & \tilde{\mathbf{B}}(\cdot) \end{bmatrix} \begin{bmatrix} \mathbf{F} \\ E \\ T \end{bmatrix} = \begin{bmatrix} q_F \\ q_E \\ q_T \end{bmatrix}, \quad (12a)$$

where  $\mathbf{M}_F, \mathbf{M}_E$ , and  $\mathbf{M}_a$  are mass matrices with coefficients  $\tilde{\sigma}, c\tilde{\sigma}$ , and  $\sigma$ , respectively,  $\mathbf{D}$  is the divergence operator, and

$$u^T \mathbf{B}(T) = \int u \sigma B(T) \, d\mathbf{x}, \quad u \in Y_0, \quad (12b)$$

$$u^T \tilde{\mathbf{B}}(T) = \int u \left( \frac{C_v}{\Delta t} + \sigma B(\cdot) \right) T \, d\mathbf{x}, \quad u \in Y_0, \quad (12c)$$

are the discrete emission operators. The source terms  $q_F, q_E$ , and  $q_T$  contain the discretized boundary condition, time-edge sources, and SMM correction sources. This system is solved with a Newton-type iteration where temperature is nonlinearly eliminated locally at each point in space.

For the nonlinear equation  $f(u) = b$ , Newton's method is,

$$f(u_0) + \mathbf{J}(u_0)(u - u_0) = b,$$

where  $u_0$  is the initial guess or previous solution iterate and  $\mathbf{J}$  is the Jacobian of  $f$ . Newton's method applied to [eq. \(12a\)](#) is,

$$\begin{bmatrix} \mathbf{M}_F & -\frac{\epsilon}{3}\mathbf{D}^T \\ \mathbf{D} & \mathbf{M}_E \\ & -\mathbf{M}_a \end{bmatrix} \begin{bmatrix} \mathbf{F}_0 \\ E_0 \\ T_0 \end{bmatrix} + \begin{bmatrix} \mathbf{M}_F & -\frac{\epsilon}{3}\mathbf{D}^T \\ \mathbf{D} & \mathbf{M}_E \\ & -\mathbf{M}_a \end{bmatrix} \begin{bmatrix} \mathbf{F} - \mathbf{F}_0 \\ E - E_0 \\ T - T_0 \end{bmatrix} = \begin{bmatrix} q_F \\ q_E \\ q_T \end{bmatrix} \quad (13a)$$

where  $\delta\mathbf{B}$  and  $\delta\tilde{\mathbf{B}}$  are linearized emission operators assembled from the bilinear forms,

$$u^T \delta\mathbf{B}v = \int \sigma \frac{\partial}{\partial T} B(T)|_{T=T_0} uv \, d\mathbf{x} = 4ac \int \sigma T_0^3 uv \, d\mathbf{x}, \quad u, v \in Y_0, \quad (13b)$$

$$u^T \delta\tilde{\mathbf{B}}v = \int \left( \frac{C_v}{\Delta t} + 4ac\sigma T_0^3 \right) uv \, d\mathbf{x}, \quad u, v \in Y_0. \quad (13c)$$

The linearity of the operators associated with the radiation energy density and radiation flux allows us to simplify [eq. \(13a\)](#) to,

$$\begin{bmatrix} \mathbf{M}_F & -\frac{\epsilon}{3}\mathbf{D}^T \\ \mathbf{D} & \mathbf{M}_E \\ & -\mathbf{M}_a \end{bmatrix} \begin{bmatrix} \mathbf{F} \\ E \\ T - T_0 \end{bmatrix} = \begin{bmatrix} q_F \\ q_E + \mathbf{B}(T_0) \\ q_T - \tilde{\mathbf{B}}(T_0) \end{bmatrix}. \quad (14)$$

#### 2.4.3. Temperature elimination

Because  $\delta\tilde{\mathbf{B}}$  is block-diagonal-by-element and can be locally inverted on each element, we may eliminate the temperature dependence in the moment system by “solving” the third row of [eq. \(14\)](#) for  $T - T_0$ ,

$$T - T_0 = (\delta\tilde{\mathbf{B}})^{-1}(q_T - \tilde{\mathbf{B}}(T_0) + \mathbf{M}_a E),$$

and substituting this equation into the second row of [eq. \(14\)](#). With temperature eliminated, the  $2 \times 2$  block system for  $E$  and  $\mathbf{F}$  is,

$$\begin{bmatrix} \mathbf{M}_F & -\frac{\epsilon}{3}\mathbf{D}^T \\ \mathbf{D} & \mathbf{M}_E - \delta\mathbf{B}(\delta\tilde{\mathbf{B}})^{-1}\mathbf{M}_a \end{bmatrix} \begin{bmatrix} \mathbf{F} \\ E \end{bmatrix} = \begin{bmatrix} q_F \\ q_E + \mathbf{B}(T_0) + \delta\mathbf{B}(\delta\tilde{\mathbf{B}})^{-1}(q_T - \tilde{\mathbf{B}}(T_0)) \end{bmatrix}.$$

To place this in the context of the  $3 \times 3$  block system, consider [eq. \(14\)](#) with rows 1 and 2 swapped,

$$\begin{bmatrix} \mathbf{M}_E & \mathbf{D} & -\delta\mathbf{B}(\cdot) \\ -\frac{\epsilon}{3}\mathbf{D}^T & \mathbf{M}_F & \\ -\mathbf{M}_a & & \delta\tilde{\mathbf{B}}(\cdot) \end{bmatrix} \begin{bmatrix} E \\ \mathbf{F} \\ T - T_0 \end{bmatrix} = \begin{bmatrix} q_E + \mathbf{B}(T_0) \\ q_F \\ q_T - \tilde{\mathbf{B}}(T_0) \end{bmatrix}, \quad (15)$$

and introduce an upper triangular temperature-elimination matrix,

$$\begin{bmatrix} I & & \delta\mathbf{B}(\delta\tilde{\mathbf{B}})^{-1} \\ & I & \\ & & I \end{bmatrix},$$

where  $I$  is the identity block matrix. Left-multiplying both sides of eq. (15) by the elimination matrix gives,

$$\begin{bmatrix} \mathbf{M}_E - \delta\mathbf{B}(\delta\tilde{\mathbf{B}})^{-1}\mathbf{M}_a & \mathbf{D} \\ -\frac{\epsilon}{3}\mathbf{D}^T & \mathbf{M}_F \\ -\mathbf{M}_a & \delta\tilde{\mathbf{B}}(\cdot) \end{bmatrix} \begin{bmatrix} E \\ \mathbf{F} \\ T - T_0 \end{bmatrix} = \begin{bmatrix} q_E + \mathbf{B}(T_0) + \delta\mathbf{B}(\delta\tilde{\mathbf{B}})^{-1}(q_T - \tilde{\mathbf{B}}(T_0)) \\ q_F \\ q_T - \tilde{\mathbf{B}}(T_0) \end{bmatrix}.$$

This linear system can be solved by first solving a  $2 \times 2$  system for  $E$  and  $\mathbf{F}$ ,

$$\begin{bmatrix} \mathbf{M}_E - \delta\mathbf{B}(\delta\tilde{\mathbf{B}})^{-1}\mathbf{M}_a & \mathbf{D} \\ -\frac{\epsilon}{3}\mathbf{D}^T & \mathbf{M}_F \end{bmatrix} \begin{bmatrix} E \\ \mathbf{F} \end{bmatrix} = \begin{bmatrix} q_E + \mathbf{B}(T_0) + \delta\mathbf{B}(\delta\tilde{\mathbf{B}})^{-1}(q_T - \tilde{\mathbf{B}}(T_0)) \\ q_F \end{bmatrix}. \quad (16)$$

The  $2 \times 2$  block system exploits the structure of  $\delta\tilde{\mathbf{B}}$ , which is block-diagonal-by-element and can be locally inverted on each element. Additionally, hybridization can be used with the  $2 \times 2$  block system to form a smaller system for Lagrange multipliers that enforce the continuity of the normal component of the radiation flux. The coefficient matrix for the Lagrange multiplier system is symmetric positive definite, so it can be efficiently inverted using conjugate gradient after algebraic multigrid preconditioning.

After computing  $E$  and  $\mathbf{F}$ , we could use forward substitution to solve the last equation in the  $3 \times 3$  block system for  $T$ ,

$$T = T_0 + (\delta\tilde{\mathbf{B}})^{-1}(q_T - \tilde{\mathbf{B}}(T_0) + \mathbf{M}_a E). \quad (17)$$

However, numerical evidence collected by Olivier et al. (2026) suggests that this update can overshoot the temperature. We follow the procedure used in Olivier et al. (2026) and apply nonlinear elimination to nonlinearly solve the energy balance equation given a radiation energy density. That is, we solve,

$$\tilde{\mathbf{B}}(T) = \mathbf{M}_a E + q_T, \quad (18)$$

using a nested Newton iteration. This nonlinear solve corresponds to solving row three of eq. (12a), which is the discrete system of nonlinear equations to which we applied the Newton-type method. The nested Newton iteration is equivalent to repeatedly solving eq. (17) swapping  $T \rightarrow T_0$  until the temperature converges and satisfies eq. (18) to iterative tolerances. Figure 2 shows the iteration between the linear solve for  $E$  and  $\mathbf{F}$  and the element-local nonlinear solve for  $T$ . Figure 3 shows the three levels of iteration: outer, inner, and nested Newton.

#### 2.4.4. Reflecting boundary condition

In addition to the inflow boundary condition in eq. (1c), we also consider a reflecting boundary condition. In the Monte Carlo component of the method, a reflecting boundary condition is achieved by simply changing the direction of the Monte Carlo particles that collide with the boundary. Reflection is achieved in the system of angle-independent equations by replacing the boundary condition in eq. (4c) with,

$$\mathbf{F} \cdot \mathbf{n} = 0, \quad \mathbf{x} \in \partial\mathcal{D}. \quad (19)$$

How do we enforce this boundary condition? First, consider how we enforced the inflow boundary condition. We incorporated eq. (4c) into the weak form using the integration

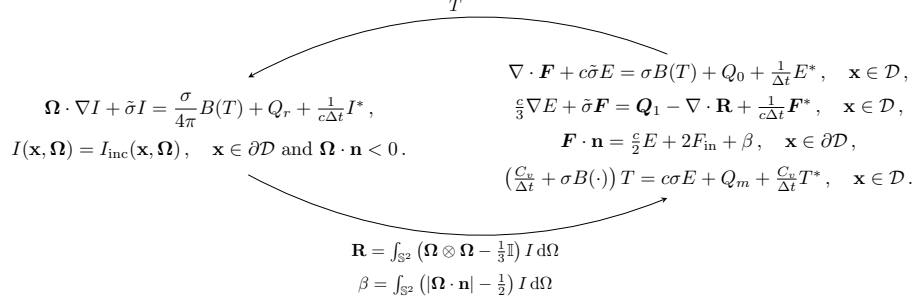


**Figure 1.** Gray TRT Second Moment outer iteration.

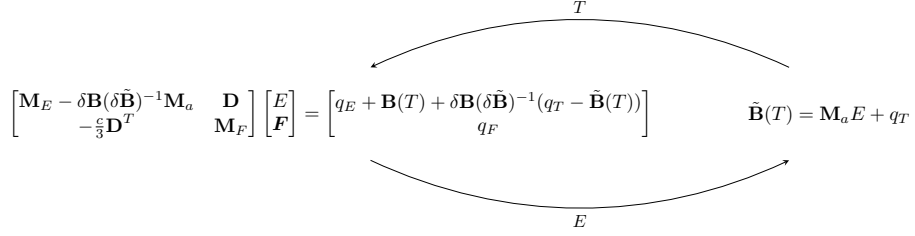


**Figure 2.** Gray TRT Second Moment inner iteration.

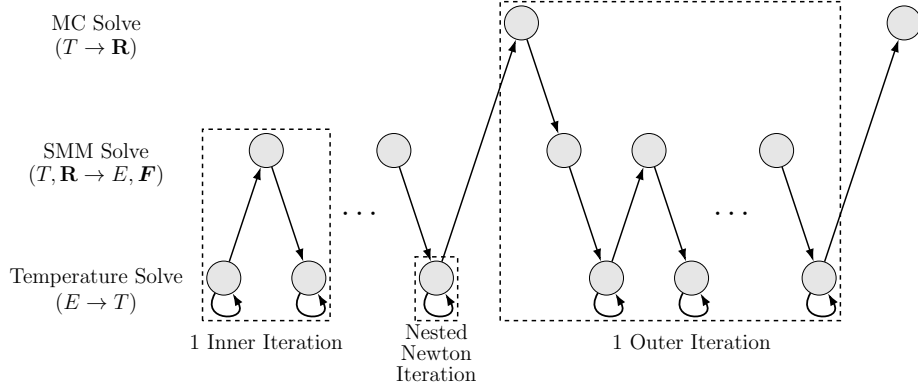


**Figure 3.** The three levels of iteration in the gray TRT Second Moment method.

by parts rule,

$$\int \frac{c}{3} \nabla E \cdot \mathbf{v} \, d\mathbf{x} = \frac{c}{3} \int_{\Gamma_b} E(\mathbf{v} \cdot \mathbf{n}) \, ds - \frac{c}{3} \int E \nabla \cdot \mathbf{v} \, d\mathbf{x}, \quad (20)$$

where we used  $\nabla \cdot (E\mathbf{v}) = \nabla E \cdot \mathbf{v} + E \nabla \cdot \mathbf{v}$  and  $\int \nabla \cdot (E\mathbf{v}) \, d\mathbf{x} = \int_{\Gamma_b} E(\mathbf{v} \cdot \mathbf{n}) \, ds$ . “Solving” eq. (4c) for  $E$  and substituting it into the boundary integral in eq. (20) generates boundary terms, one on the left-hand side (LHS) and one on the right-hand side (RHS) of the resulting equation:

$$\text{LHS term: } \frac{2}{3} \int_{\Gamma_b} (\mathbf{v} \cdot \mathbf{n})(\mathbf{F} \cdot \mathbf{n}) \, ds \quad \text{RHS term: } \frac{2}{3} \int_{\Gamma_b} (\mathbf{v} \cdot \mathbf{n})(2F_{\text{in}} + \beta) \, ds.$$

In the  $3 \times 3$  block system of discrete nonlinear equations in (12a), the LHS term contributes to the  $\mathbf{M}_F$  block and the RHS term to the  $q_F$  vector. Incorporating the reflecting boundary condition eq. (19) instead of the inflow boundary condition results in a weak form without the LHS and RHS terms. After applying the integration by parts rule in eq. (20), we force the boundary integral to be zero by strongly enforcing the boundary condition. That is, we incorporate the boundary condition in the trial function space definition. We do this by setting the solution vector to zero and eliminating the boundary degrees of freedom so that they do not appear in the linear system. To clarify, consider the  $2 \times 2$  system for  $E$  and  $\mathbf{F}$  in eq. (16), rewritten here with asterisks (\*) where the system is unaffected by the reflecting boundary condition,

$$\begin{pmatrix} \mathbf{M}_F & * \\ \mathbf{D} & * \end{pmatrix} \begin{pmatrix} \mathbf{F} \\ E \end{pmatrix} = \begin{pmatrix} q_F \\ * \end{pmatrix}. \quad (21)$$

In addition to modifying the  $\mathbf{M}_F$  block and  $q_F$  vector to remove the contribution of the LHS and RHS boundary integrals, we eliminate essential columns in  $\mathbf{D}$ , essential entries in  $q_F$ , and both rows and columns in  $\mathbf{M}_F$ .

### 2.5. Monte Carlo component of the hybrid method

The Boltzmann equation on the left-hand side of Figure 1 depends on space and angle. Unphysical patterns caused by angular under-resolution, known as “ray effects”, can be avoided by using Monte Carlo to solve the equation because Monte Carlo does not require angular discretization. However, Monte Carlo introduces undesirable statistical variation, hereafter referred to as “noise”. The noise problem in the Monte Carlo Second Moment linear transport method of Pozulp et al. (2026) required variance reduction. Our numerical results suggest that variance reduction is also required for TRT.

An important part of the Monte Carlo solve is estimating the data  $\mathbf{R}$  and  $\beta$  for the system of angle-independent equations on the right-hand side of Figure 1. Distributing the integrals in the functionals defining  $\mathbf{R}$  and  $\beta$  results in,

$$\mathbf{R} = \int_{\mathbb{S}^2} \boldsymbol{\Omega} \otimes \boldsymbol{\Omega} I \, d\Omega - \frac{1}{3} \mathbb{I} \int_{\mathbb{S}^2} I \, d\Omega, \quad (22)$$

$$\beta = \int_{\mathbb{S}^2} |\boldsymbol{\Omega} \cdot \mathbf{n}| I \, d\Omega - \frac{1}{2} \int_{\mathbb{S}^2} I \, d\Omega, \quad (23)$$

where  $\mathbb{I}$  is the identity tensor. We estimate  $\mathbf{R}$  using path length estimators as,

$$\hat{\mathbf{R}} = c\hat{\mathbf{P}} - \frac{1}{3} \mathbb{I} c\hat{E}, \quad (24)$$

where  $\hat{E}$  and  $\hat{\mathbf{P}}$  are path length estimators for the radiation energy density and radiation pressure,

$$c\hat{E} = \frac{1}{\text{vol}(K)\Delta t} \sum_i d_i w_i, \quad (25)$$

$$c\hat{\mathbf{P}} = \frac{1}{\text{vol}(K)\Delta t} \sum_i \boldsymbol{\Omega}_i \otimes \boldsymbol{\Omega}_i d_i w_i. \quad (26)$$

In eqs. (25) and (26),  $\text{vol}(K)$  is the volume of mesh element  $K$ , the sums are over all the particle paths traversed in element  $K$ ,  $d_i$  is the distance of path  $i$ ,  $w_i$  is the weight of the particle that traversed path  $i$ , and  $\boldsymbol{\Omega}_i$  is the direction of the particle that traversed path  $i$  (not the  $i$ th entry of  $\boldsymbol{\Omega}$  but rather the whole vector). We estimate  $\beta$  using surface crossing estimators as,

$$\hat{\beta} = \hat{B} - \frac{1}{2}c\hat{E}_s, \quad (27)$$

where  $\hat{B}$  and  $\hat{E}_s$  are,

$$\hat{B} = \frac{1}{\text{area}(\mathcal{F})\Delta t} \sum_i w_i, \quad (28)$$

$$c\hat{E}_s = \frac{1}{\text{area}(\mathcal{F})\Delta t} \sum_i \frac{w_i}{|\boldsymbol{\Omega}_i \cdot \mathbf{n}|}. \quad (29)$$

In eqs. (28) and (29),  $\text{area}(\mathcal{F})$  is the area of the boundary mesh face  $\mathcal{F}$  and the sums are over all the particle paths that cross the domain boundary. This includes not only particles that leave the domain but also particles that enter (from the inflow source).

The need for variance reduction can be understood through a thick diffusion limit analysis:  $E$  and  $\mathbf{P}$  are  $O(1)$  quantities whose contributions must cancel to yield the  $O(\varepsilon^2)$  quantity  $\mathbf{R}$  for the method to be accurate, where  $\varepsilon \in (0, 1]$  is the scaling parameter that approaches zero in the thick diffusion limit. The  $O(N^{-1/2})$  fluctuations in the Monte Carlo estimators for calculations with  $N$  particles cause statistical catastrophic cancellation if  $N$  does not grow in proportion to  $\varepsilon$  (that is, accuracy requires  $N \approx \varepsilon^{-1}$ ). A source transformation that provides analytic cancellation would mitigate this effect and provide greater accuracy for a given  $N$ .

## 2.6. Hybrid gray TRT Second Moment method

Algorithm 1 shows pseudocode for the hybrid gray TRT Second Moment method described thus far. The first Monte Carlo transport solve of every time step uses the fixed source functions  $Q_r$ ,  $I_{\text{inc}}$ , and  $I^*$ , whereas the subsequent Monte Carlo transport solves in the same time step use the emission source function computed with the most recent temperature iterate,  $B(T^{(i)})$ . The outer iteration converges when the  $L^2(\mathcal{D})$  norm of the difference of successive temperature iterates falls below a user-provided threshold  $\eta_{\text{outer}}$ ,

$$\|T^{(i)} - T^{(i-1)}\|_2 < \eta_{\text{outer}}. \quad (30)$$

The inner iteration converges when the relative difference of successive radiation energy density iterates falls below a user-provided threshold  $\eta_{\text{inner}}$ ,

$$\frac{\|E^{(j)} - E^{(j-1)}\|_2}{\|E^{(j)}\|_2} < \eta_{\text{inner}}, \quad (31)$$

where  $j$  is the inner iteration index. The nested Newton iteration on [line 12](#) converges when the relative difference of successive temperature iterates falls below a user-provided threshold  $\eta_{\text{nested\_newton}}$

$$\frac{\|T^{(k)} - T^{(k-1)}\|_2}{\|T^{(k)}\|_2} < \eta_{\text{nested\_newton}}, \quad (32)$$

where  $k$  is the nested Newton iteration index.

---

**Algorithm 1** Hybrid gray TRT Second Moment method.

---

```

1: while time < final time do
2:   Compute time-edge sources and opacity  $\tilde{\sigma}$ 
3:    $T^{(0)} \leftarrow T^*$ 
4:    $\hat{\mathbf{R}}^{(0)}, \hat{\beta}^{(0)} \leftarrow$  Solve eq. \(7a\) subject to eq. \(7b\) with  $B(T) = 0$  and
       $(Q_r, I_{\text{inc}}, I^*) \neq 0$  using Monte Carlo
5:    $i \leftarrow 1$ 
6:   while outer not converged do
7:      $j \leftarrow 1$ 
8:     while inner not converged do
9:       Assemble Jacobian
10:      Eliminate temperature dependence
11:       $E^{(j)}, \mathbf{F}^{(j)} \leftarrow$  Solve eq. \(16\) using FEM with data
         $(Q_0, \mathbf{Q}_1, F_{\text{in}}, E^*, \mathbf{F}^*, \hat{\mathbf{R}}^{(i-1)}, \hat{\beta}^{(i-1)}, T^{(j-1)})$ 
12:       $T^{(j)} \leftarrow$  Nonlinearly solve eq. \(18\) with data  $(Q_m, T^*, E^{(j)})$ 
13:       $j \leftarrow j + 1$ 
14:    end while
15:     $T^{(i)} \leftarrow T^{(j-1)}$ 
16:     $\hat{\mathbf{R}}_{\text{temp}}, \hat{\beta}_{\text{temp}} \leftarrow$  Solve eq. \(7a\) subject to eq. \(7b\) with  $B(T^{(i)}) \neq 0$  and
       $(Q_r, I_{\text{inc}}, I^*) = 0$  using Monte Carlo
17:     $\hat{\mathbf{R}}^{(i)} \leftarrow \hat{\mathbf{R}}^{(0)} + \hat{\mathbf{R}}_{\text{temp}}$ 
18:     $\hat{\beta}^{(i)} \leftarrow \hat{\beta}^{(0)} + \hat{\beta}_{\text{temp}}$ 
19:     $i \leftarrow i + 1$ 
20:  end while
21: end while

```

---

### 3. Numerical results

We implemented our method in 2D-XY geometry, so the phase space of our gray TRT calculations is five dimensional: two space dimensions plus two angle dimensions and one time dimension. We wrote an MPI-parallel Monte Carlo solver in the C programming language that distributes the simulation particles evenly among the MPI ranks. All ranks source and track particles. To implement the deterministic solvers, we used the C++ programming language with the MPI-parallel finite element library MFEM ([Anderson et al. 2021](#)) along with the MPI-parallel sparse direct linear solver library SuperLU ([Li 2005](#)). We used SuperLU to invert the  $2 \times 2$  block system for

$E$  and  $\mathbf{F}$  in eq. (16) and to perform every linear solve for  $T$  in the nested Newton iteration of the material energy balance eq. (18). We used SuperLU because, in the largest problem we considered, linear solve time accounted for less than 5% of the total calculation runtime. This allowed us the luxury of computing the solution to (a small multiple of) machine precision. However, SuperLU is not required; fast iterative linear solvers can be used to invert eq. (16) and have been shown to be efficient on large systems (Olivier and Haut 2024). We set the user-defined convergence thresholds in eqs. (30) to (32) to,

$$(\eta_{\text{outer}}, \eta_{\text{inner}}, \eta_{\text{nested\_newton}}) = (10^{-3}, 10^{-6}, 10^{-10}).$$

For some problems, we compare to solutions calculated by solving Fleck and Cummings' linearized TRT equations using Fleck and Cummings' Monte Carlo technique for solving the linearized equations (Fleck and Cummings 1971). Applying Fleck and Cummings' linearization to the gray TRT system of eqs. (1a) and (1b) gives,

$$\frac{1}{c} \frac{\partial I}{\partial t} + \boldsymbol{\Omega} \cdot \nabla I + \sigma I = \frac{1-f}{4\pi} \sigma \int_{\mathbb{S}^2} I \, d\boldsymbol{\Omega} + \frac{f}{4\pi} c \sigma U(T) + \frac{1-f}{4\pi} Q_m + Q_r, \quad (33a)$$

$$C_v \frac{\partial T}{\partial t} = f c \sigma E - f c \sigma U(T) + f Q_m, \quad (33b)$$

where  $U(T)$  is a nonlinear function of the temperature,

$$U(T) = B(T)/c = aT^4,$$

$f$  is the Fleck factor,

$$f = \frac{1}{1 + \alpha \beta' \sigma c \Delta t}, \quad (33c)$$

where  $\alpha \in [0, 1]$  is the implicitness factor and  $\beta'$  is the strength of the coupling between the material and the radiation,

$$\beta' = \beta'(\mathbf{x}, t) = \frac{\partial U(T)}{\partial U_m} = \frac{dU(T)/dT}{dU_m/dT} = \frac{4aT^3}{C_v}. \quad (33d)$$

The coupling term is most often denoted  $\beta$  instead of  $\beta'$ , but we use the latter because we already used the former to denote the boundary correction factor in eq. (4c). We set  $\alpha = 1$  in all calculations where we present Fleck and Cummings results.

### 3.1. Method of manufactured solutions problem

The method of manufactured solutions (MMS) is a code verification technique in which one chooses an analytic solution, which is called the ‘‘MMS solution’’. One then substitutes the MMS solution into the partial differential equation to derive expressions for the problem data, which include the sources, boundary conditions, and initial condition (Roache 1998; Salari and Knupp 2000; Roache 2001). We use MMS to examine whether our method has a numerical error consistent with  $O(\Delta t) + O(N^{-1/2}) + O(h)$ , where  $\Delta t$  is the time step size,  $N$  is the number of Monte Carlo particles per time step, and  $h$  is the characteristic mesh length defined by,

$$h = \max_{K \in \mathcal{T}} \left( \int_{\hat{K}} J \, d\boldsymbol{\xi} \right)^{1/\dim}, \quad (34)$$

where the maximum is over all elements  $K$  in the mesh  $\mathcal{T}$ , the integration is on the reference element  $\hat{K}$ ,  $J$  is the Jacobian determinant of the mapping from the reference

element to the physical element, and  $\xi$  denotes the reference-space coordinates. The  $O(\Delta t)$  term is due to the semi-implicit backward Euler time integration. The  $O(h)$  term is due to the  $h^{p+1}$  convergence of the finite element discretization of the second moment system, where the basis function polynomial order  $p = 0$  in our case because we use  $Y_0$ ,  $RT_0$ , and piecewise constant Monte Carlo estimators. The  $O(N^{-1/2})$  term is due to the  $\sqrt{\text{Var}[\cdot]/N}$  uncertainty of the estimators computed in the Monte Carlo component of our method, where  $\text{Var}[\cdot]$  is the variance of some estimator.

If the time step size  $\Delta t$  is decreased such that the size of the  $O(\Delta t)$  term is nearly zero, then the hypothesized convergence rate due to the remaining terms,  $O(N^{-1/2}) + O(h)$ , defines an error surface that decreases in height as one traverses simultaneously upward and rightward in Figure 4. By making  $h$  small and running multiple calculations with increasing  $N$  in the “particle scaling study”, we observe the  $O(N^{-1/2})$  term in the hypothesized convergence rate, and by making  $N$  large and running multiple calculations with decreasing  $h$  in the “element scaling study”, we observe the  $O(h)$  term.

The MMS solution that we compute is from Yee et al. (2021) and the problem data is from Brunner (2006). The material has a constant absorption opacity  $\sigma = 4 \text{ m}^{-1}$  and a constant heat capacity  $C_v = 6 \text{ J / (m}^3 \text{ K)}$ . The initial temperature of the domain is  $T_0 = 10,000 \text{ K}$  ( $\approx 0.86 \text{ eV}$ ). The MMS solution is periodic in space and exponentially decaying in time. The decay rate  $\tau$  is,

$$\tau = \frac{241c\sigma a T_0^3}{16C_v}, \quad (35)$$

where  $c$  is the speed of light and  $a$  is the constant,

$$a = \frac{8\pi^5 k^4}{15h^3 c^3}. \quad (36)$$

The angular frequency  $\omega$  of the periodic part of the solution is,

$$\omega = \sqrt{\frac{15\tau\sigma}{c}}. \quad (37)$$

The MMS solution is,

$$I_{\text{MMS}}(x, y, \mathbf{\Omega}, t) = \frac{1}{4\pi} \left[ \varphi(x, y, t) \theta(\mathbf{\Omega}) + \varphi_1(x, y, t) \theta_1(\mathbf{\Omega}) \right], \quad (38a)$$

$$T_{\text{MMS}}(x, y, t) = T_0 \left[ 1 - \frac{e^{-\tau t}}{2} \cos(\omega x) \cos(\omega y) \right], \quad (38b)$$

where  $\varphi, \varphi_1, \theta$ , and  $\theta_1$  are,

$$\varphi(x, y, t) = acT_{\text{rad}}^4 \left[ 1 + \frac{e^{-\tau t}}{2} \cos(\omega x) \cos(\omega y) \right], \quad (38c)$$

$$\varphi_1(x, y, t) = acT_{\text{rad}}^4 \frac{e^{-\tau t}}{2} \sin(\omega x) \sin(\omega y), \quad (38d)$$

$$\theta(\mathbf{\Omega}) = \frac{1}{2} + \frac{3}{2} \Omega_z^2, \quad (38e)$$

$$\theta_1(\mathbf{\Omega}) = \frac{1}{3} \Omega_x \Omega_y. \quad (38f)$$

Finally,  $T_{\text{rad}}$  in eqs. (38c) and (38d) is,

$$T_{\text{rad}}(t) = T_0 \left( 1 + \frac{e^{-\tau t}}{2} \right). \quad (38g)$$

The domain is the square defined by  $0 \leq x, y \leq 3\pi\omega^{-1}$ . We run for a total simulation time of 1 ns. We use a time step size of  $\Delta t = 0.5$  ns to run two time steps for the calculation with the fewest particles in the particle scaling study,  $(N, h^{-1}) = (4^8, 2^6)$ , and for the calculation with the fewest elements in the element scaling study,  $(N, h^{-1}) = (4^{13}, 2^2)$ . We decrease the time step size by a factor of two when we move right one point in the particle scaling study, and when we move up one point in the element scaling study. Thus, the number of time steps is: 2, 4, 8, 16, and 32 as we go left-to-right and bottom-to-top in Figure 4. We use a computational mesh of equally-sized squares and we compute the  $L^2(\mathcal{D})$  error between our numerical solution and the MMS solution at 1 ns,  $T_{\text{MMS}}(x, y, t = 1 \text{ ns})$ . Figure 5 shows a plot of  $T_{\text{MMS}}(x, y, t = 1 \text{ ns})$  on the calculation domain  $0 \leq x, y \leq 3\pi\omega^{-1}$ .

The results in Figure 6 indicate that our method converges under mesh refinement and Monte Carlo sample augmentation, though the rate of convergence with respect to the element width and number of Monte Carlo particles does not exactly match our hypothesis of  $O(h)$  and  $O(N^{-1/2})$ , respectively. The hypothesized value of the slopes of the line segments in Figure 6a is -0.5, yet the actual slopes are -0.34, -0.39, -0.65, and -0.48. In Figure 6b, the hypothesized value of the slopes of the line segments is unity, yet the actual slopes are 0.65, 0.82, 0.90, and 0.95. Slope values approaching the hypothesized values as the Monte Carlo sample size increases and the mesh is refined suggest that we are correctly solving the gray TRT equations.

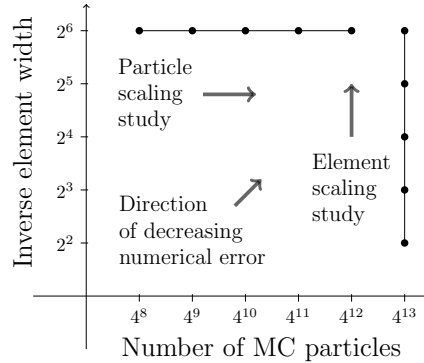


Figure 4. Calculation points on the  $O(N^{-1/2}) + O(h)$  error surface.

### 3.2. Infinite medium problem

An infinite medium has a constant temperature for all time, so this problem exercises the ability of the numerical method to maintain the initial temperature. We model an infinite medium in a  $1 \text{ cm} \times 1 \text{ cm}$  square domain with reflecting boundaries using a  $10 \times 10$  mesh of equal-sized square elements, and measure temperature at a point. Figure 7 shows the domain, boundary conditions, mesh, and temperature measurement point. Enforcing  $\mathbf{F} \cdot \mathbf{n} = 0$ , as described in Section 2.4.4, is how we achieve reflection in

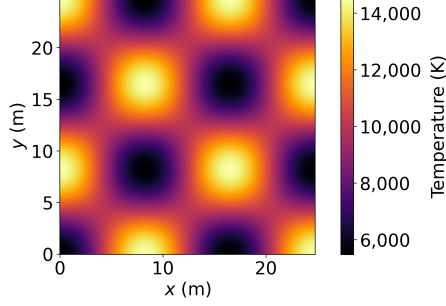


**Figure 5.** The MMS solution  $T_{\text{MMS}}(x, y, t = 1 \text{ ns})$  on the domain  $0 \leq x, y \leq 3\pi\omega^{-1}$ .

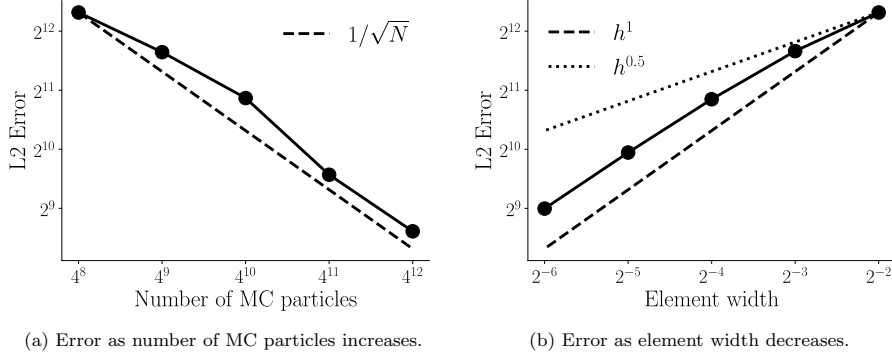


**Figure 6.** Error of iterate  $T^{(i)}$  upon convergence with respect to the MMS solution  $\bar{T}_{\text{MMS}}$  at 1 ns.

the deterministic component of the method. Reflection in the Monte Carlo component is achieved by simply changing the direction of the Monte Carlo particles that collide with the boundary, as depicted by the arrows in [Figure 7](#).

The infinite medium is initialized with a temperature of 1 keV. It has a constant absorption opacity  $\sigma = 1 \text{ cm}^{-1}$  and a constant heat capacity  $C_v = 0.1 \text{ GJ} / (\text{cm}^3 \text{ keV})$ . We compute 100 time steps with a time step size of  $\Delta t = 200 \text{ ps}$  for a total simulation time of 20 ns. We run 100,000 Monte Carlo particles per time step. In addition to the temperature, which is a measure of the amount of energy in the material, we also plot the radiation temperature, which is a measure of the amount of energy in the radiation field. The radiation temperature is the  $T_{\text{rad}}$  that satisfies the equation,

$$E = aT_{\text{rad}}^4, \quad (39)$$

where  $E$  is the radiation energy density and  $a$  is the constant defined by [eq. \(36\)](#).

We compare our method to the method of [Fleck and Cummings \(1971\)](#), which solves Fleck and Cummings' system of linearized gray TRT [eqs. \(33a\) to \(33d\)](#) using Monte Carlo. We use the Monte Carlo estimator  $\hat{E}$  of the radiation energy density defined in [eq. \(25\)](#) to compute the radiation temperature with Fleck and Cummings' method. For our method, we use the converged  $E^{(i)}$  iterate (see [line 11](#) in [Algorithm 1](#)) to compute the radiation temperature, rather than  $\hat{E}$  from [eq. \(25\)](#).

[Figure 8](#) shows temperature time series plots for the infinite medium calculation,

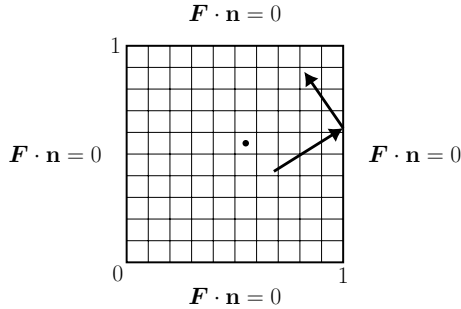


**Figure 7.** Domain, boundary conditions, mesh, and temperature measurement point for the infinite medium problem.

in which we see that both Fleck and Cummings’ method and our method seem to preserve the infinite medium temperature within some statistical error. The radiation temperature appears to follow the material temperature as it bounces around the initial value of 1 keV.

To investigate the effect of the magnitude of the emission source on the iterative convergence of our method, we increased the temperature of the infinite medium from 1 keV to 2 keV, re-ran the calculation, and found that the outer iteration did not converge. Increasing the number of Monte Carlo particles from 100,000 to 1,000,000 per time step allowed the outer iteration to converge, but [Figure 9](#) shows that the temperature time series is noisier than the time series from Fleck and Cummings’ method (using the same number of particles). We increased the temperature by another factor of two, re-ran the calculation, and found that the outer iteration failed to converge again. Increasing the number of Monte Carlo particles by another factor of ten allowed the outer iteration to converge, but [Figure 10](#) shows that our method converges to the wrong temperature.

Our method performs poorly as the temperature increases because the magnitude of the emission source in [eq. \(1a\)](#) grows without bound as the temperature increases. By contrast, the emission source in Fleck and Cummings’ linearized [eq. \(33a\)](#) is attenuated by the Fleck factor  $f \in (0, 1]$ , which decreases as temperature increases because of the  $T^3$  term in the denominator of  $f$  (see [eqs. \(33c\)](#) and [\(33d\)](#)).

Our noise issue is consistent with the theory developed by [Pozulp and Haut \(2026\)](#) for Monte Carlo source iteration. This suggests that the same type of source transformation variance reduction technique used in [Pozulp et al. \(2026\)](#) could help here.

### 3.3. Inflow problem

The inflow problem that we consider uses the inflow boundary condition to exercise the ability of the numerical method to compute maximum-principle-preserving solutions for different time step sizes. We model the problem in a  $1 \text{ cm} \times 1 \text{ cm}$  square domain with vacuum boundaries. On the left boundary there is an inflow radiation source corresponding to a Planck distribution at a temperature of 1 keV. The domain is initialized with a temperature of 0.1 keV. It has a constant absorption opacity  $\sigma = 1 \text{ cm}^{-1}$  and a constant heat capacity  $C_v = 0.1 \text{ GJ} / (\text{cm}^3 \text{ keV})$ . We run 1,000,000 Monte Carlo particles per time step for a total simulation time of 8 ns. We run one calculation with a time step size of  $\Delta t = 4 \text{ ns}$ , one with  $\Delta t = 2 \text{ ns}$ , and one with  $\Delta t = 1 \text{ ns}$ . We

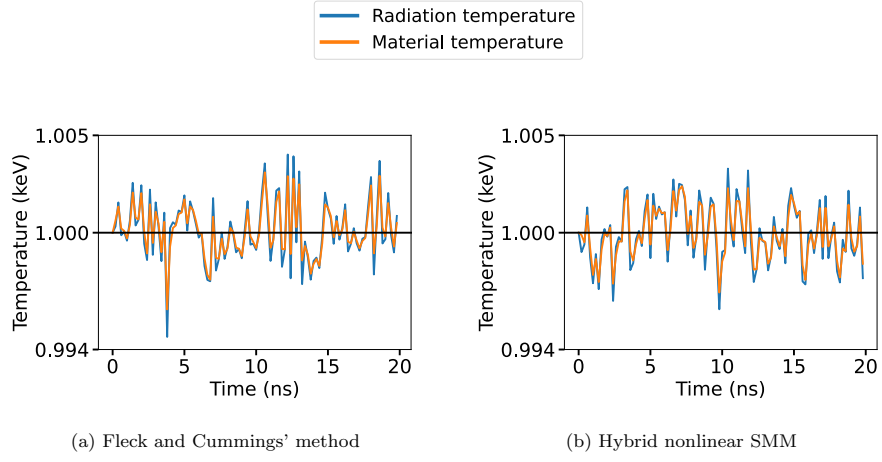


**Figure 8.** Temperature for 1 keV infinite medium using 100,000 Monte Carlo particles per time step.

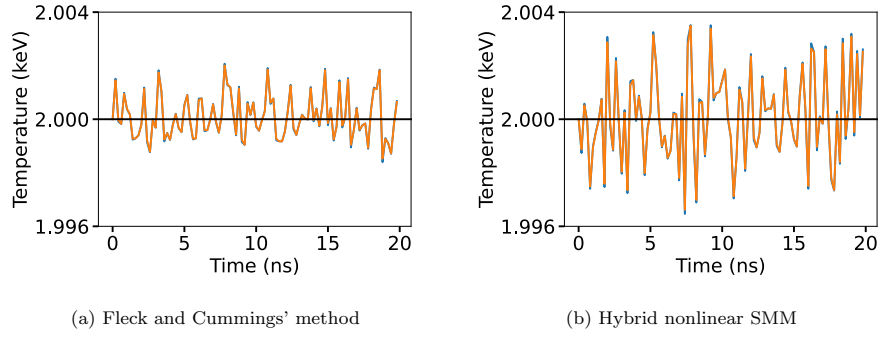


**Figure 9.** Temperature for 2 keV infinite medium using 1,000,000 Monte Carlo particles per time step.

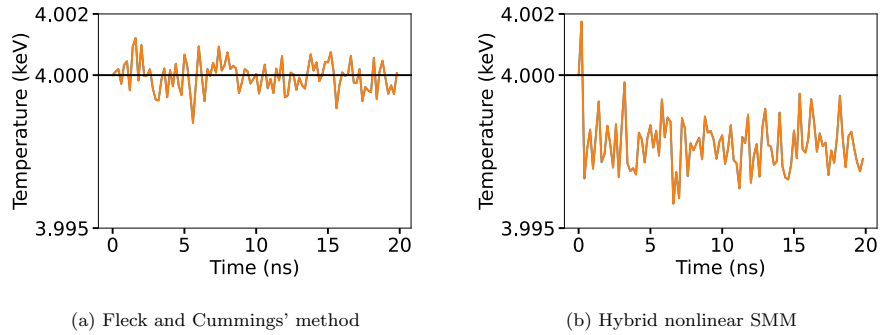


**Figure 10.** Temperature for 4 keV infinite medium using 10,000,000 Monte Carlo particles per time step.

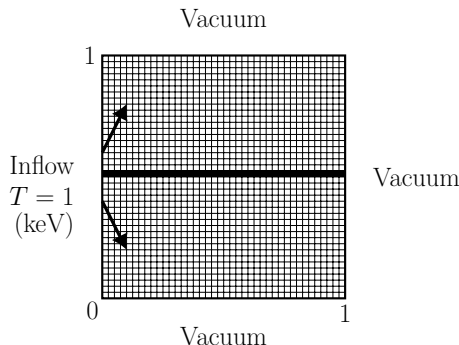


**Figure 11.** Domain, boundary conditions, mesh, and temperature measurement line for the inflow problem.

mesh the domain using a  $40 \times 40$  mesh of equal-sized square elements and measure the temperature at the end of the simulation along the horizontal line  $y = 1/2$ . Figure 11 shows the domain, boundary conditions, mesh, and temperature measurement line. The arrows in Figure 11 depict two of the directions in the hemisphere of inflow directions on the inflow boundary.

We compare our method to the method of Fleck and Cummings (1971), which solves Fleck and Cummings’ system of linearized gray TRT eqs. (33a) to (33d).

Because the inflow source is a Planck distribution at 1 keV and the domain is initialized with a temperature of 0.1 keV, solutions to the TRT equations for the inflow problem adhere to a maximum principle. To preserve it, numerical methods must compute numerical solutions in which the temperature does not exceed 1 keV anywhere on the domain. Figure 12a shows that Fleck and Cummings’ method computes temperatures that exceed the maximum allowable temperature by more than a factor of five for the 4 ns time step size. Decreasing the time step size to 2 ns reduces the overshoot, but it is not until the time step is decreased to 1 ns that the maximum principle is preserved. By contrast, Figure 12b shows that our method computes temperatures that do not violate the maximum principle for any of the three time step sizes considered.

Table 1 shows that the average number of iterations required to converge the outer, inner, and nested Newton (NN) iterations depends weakly on the time step size  $\Delta t$ .

Figure 13 shows that if we use the time step size  $\Delta t = 125$  ps to run the same calculation using 64 time steps, then Fleck and Cummings’ method and our method show better agreement. Interestingly, the temperature lineout for our method has an unphysical wavy pattern, like a low-amplitude sine wave perturbation.

Figure 14 shows pseudocolor plots of the temperature on the whole domain for Fleck and Cummings’ method and for our method, from which we can see that the waves in the lineout in our method appear to be a noise pattern with lower frequency than the very small amount of noise in the temperature calculated using Fleck and Cummings. We used grayscale for these pseudocolor plots because we found that it makes the subtle differences in the noise patterns easier to discern.

#### 4. Conclusion

The desire for Monte Carlo solutions to the equations of thermal radiative transfer that preserve a maximum principle motivated us to design a method with implicit emission-

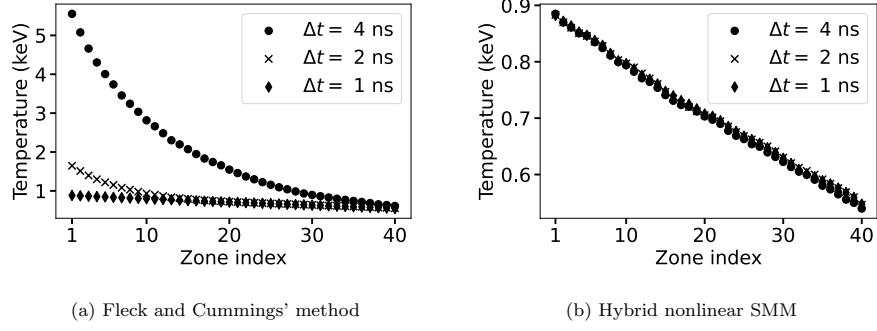


**Figure 12.** Temperature lineouts for the inflow problem.

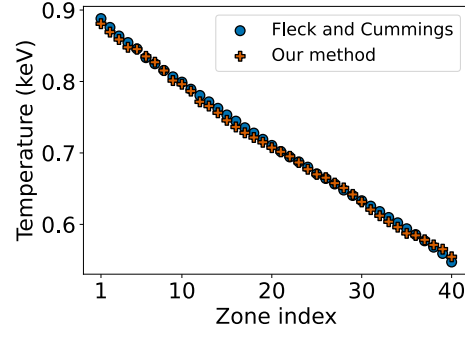


**Figure 13.** Temperature lineouts for the inflow problem with  $\Delta t = 125$  ps (64 time steps).

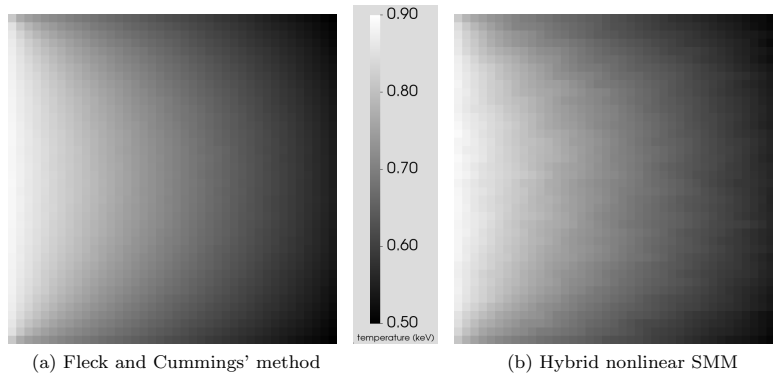


**Figure 14.** Temperature pseudocolor plots for the inflow problem.

| Timestep size $\Delta t$ (ns) |       | 4    | 2    | 1    |
|-------------------------------|-------|------|------|------|
| Number of time steps          |       | 2    | 4    | 8    |
| Outer                         | Min   | 3    | 3    | 3    |
|                               | Max   | 4    | 4    | 4    |
|                               | Avg.  | 3.50 | 3.75 | 3.88 |
|                               | Total | 7    | 15   | 31   |
| Inner                         | Min   | 2    | 2    | 2    |
|                               | Max   | 4    | 5    | 4    |
|                               | Avg.  | 2.71 | 2.93 | 2.81 |
|                               | Total | 19   | 44   | 87   |
| Nested Newton                 | Min   | 1    | 1    | 1    |
|                               | Max   | 14   | 12   | 9    |
|                               | Avg.  | 3.63 | 2.95 | 2.75 |
|                               | Total | 69   | 130  | 239  |

**Table 1.** Number of iterations required to converge by time step size for the inflow problem.

absorption physics and explicit material properties. Figure 12 shows that we were at least partially successful: in a problem where large time steps cause the method of Fleck and Cummings (which has semi-implicit emission-absorption physics) to violate a maximum principle, our method does not. However, Figures 8 to 10 show that simply increasing the magnitude of the emission source can cause the outer iteration in our method to fail to converge (due to Monte Carlo noise). Increasing the emission source increases the expense of Fleck and Cummings’ method because the Fleck factor  $f$  defined in eq. (33c) decreases as temperature increases, but it does not cause the method to fail.

These results indicate that variance reduction is required. It is perhaps not surprising that another Monte Carlo TRT method with implicit emission-absorption physics and explicit material properties, the method of Brooks (1989), resulted in the invention of a novel variance reduction technique by which the emission source is transformed into time- and space-derivative sources (Szoke and Brooks III 2005). Trying Szoke and Brooks’ variance reduction technique with our method seems to be the most promising approach for correcting the noise issue when the emission is strong.

## Authors' contributions

**Michael Pozulp:** Writing – original draft, Methodology, Software; **Terry Haut:** Writing – review & editing, Conceptualization; **Patrick Brantley:** Writing – review & editing, Supervision; **Samuel Olivier:** Writing – review & editing, Methodology.

## Artificial intelligence contributions

Large language models were used to find typos in this manuscript and bugs in the software described in [Section 3](#).

## Disclosure statement

No conflict of interest was reported by the authors.

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